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ORIGINAL PAPER

Some Existence Results on a Class of Generalized Quasilinear Schrödinger Equations with Choquard Type

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Abstract
In this paper, we study the generalized quasilinear Schrödinger equation

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

where $N \geq 3$, $0 < \alpha < N$, $\frac{2(N+\alpha)}{N} < p < \frac{2(N+\alpha)}{N-2}$, $V : \mathbb{R}^N \to \mathbb{R}$ is a potential function and I_α is a Riesz potential. Under appropriate assumptions on g and $V(x)$, we establish the existence of positive solutions and ground state solutions.

Keywords Quasilinear Schrödinger equation · Positive solutions · Ground state solutions · Choquard type

Mathematics Subject Classification 35J60 · 35J20

1 Introduction

In this work, we consider the generalized quasilinear Schrödinger equation

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

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where $N \geq 3$, $0 < \alpha < N$, $\frac{2(N+\alpha)}{N} < p < \frac{2(N+\alpha)}{N-2}$, $V : \mathbb{R}^N \to \mathbb{R}$ satisfies some suitable conditions and I_α is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\Pi^{\frac{N}{2}}2^\alpha|x|^{N-\alpha}} := \frac{A_\alpha}{|x|^{N-\alpha}},$$

and Γ is the Gamma function.

It is related with the existence of solitary wave solutions for the quasilinear Schrödinger equation:

$$i\partial_t\omega = -\Delta\omega + V(x)\omega - k(x,\omega) - l'(|\omega|^2)\omega\Delta l(|\omega|^2), \tag{1.2}$$

where $\omega : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $V : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $l : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are suitable functions. For various types of l, the quasilinear Schrödinger equation (1.2) can be transformed into models reflecting different physical phenomena. For example, in [24], let $l(s) = 1$, we can get the classical stationary semilinear Schrödinger equation. If $l(s) = s$, we can see in [16,19,22,29] that the equation was acquired by fluid mechanics, plasma physics, and dissipative quantum mechanics. For more background on physics, we can refer to [2,20,23] and references therein.

Set $z(t,x) = exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and u is a real function. Equation (1.2) can be reduced to the corresponding equation of elliptic type (see [3]):

$$-\Delta u + V(x)u - \Delta l(u^2)l'(u^2)u = h(x,u), \ x \in \mathbb{R}^N. \tag{1.3}$$

If we take $g^2(u) = 1 + \frac{[(l^2(u))']^2}{2}$, then Eq. (1.3) can be written as quasilinear elliptic equations (see [32]):

$$-\text{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x,u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

As we all know, there are many papers focusing on problem (1.4) and studying the existence of standing wave solutions for Eq. (1.4) (see [4,32–34]). More specifically, in [8,9], Deng et al. studied the existence of nodal solutions with variational argument. Deng et al. [10] found the critical exponents for problem (1.4) and then considered the existence of positive solutions to Eq. (1.4) with critical exponents. In [11], Furtado studied the existence of solution in the Orlicz-Sobolev space for problem (1.4) by using the change of variables and variational argument. What's more, Eq. (1.4) was extended to include positive parameter and critical exponents, then Chen et al. [7] proved the existence and asymptotic behavior of standing wave solutions for the equation. In the previous articles, most of the authors usually think about a huge class of nonlinearities g.

In particular, if we set $g(u) = \sqrt{1+2u^2}$, Eq. (1.4) can be transformed into the following equations:

$$-\Delta u + V(x)u - \Delta(u^2)u = h(x,u), \ x \in \mathbb{R}^N. \tag{1.5}$$

The existence of a positive ground state solution for problem (1.5) was first proved by Poppenberg et al in [30]. Then, Liu and Wang [24] studied the existence of a solution of the equation with unknown Lagrange multiplies λ in front of the nonlinear term using a constrained minimization argument. Furthermore, by a change of variables, Eq. (1.5) becomes a semilinear problem and the existence of it is positive solution in Orlicz space was obtained by using the Mountain-Pass theorem in [25].

In the previous papers, the authors related the existence of weak solutions of the problem to the the critical point of the energy functional by limiting some growth restrictions on h, then we can obtain solutions for a large class of nonlinearities h by theoretical mechanism of critical points. For Eq. (1.5), if we set $h(x,u) = (I_\alpha * |u|^p)|u|^{p-2}u$, then it becomes

$$-\Delta u + V(x)u - \Delta(u^2)u = (I_\alpha * |u|^p)|u|^{p-2}u,\ \ x \in \mathbb{R}^N. \tag{1.6}$$

To our knowledge, the Eq. (1.6) mentioned above is usually called quasilinear Schrödinger equation with Choquard type. According to nonlinear Choquard equation, it first appeared in S. I. Pekar [31]'s work. Later, Moroz and Van Schaftingen [26] studied the existence, qualitative properties and decay asymptotics of the ground state solutions for nonlinear Choquard equation. Moreover, for more articles about Choquard equation, we can refer to [14,15,28]. Recently, Chen et al. [5] proved the existence of positive solutions and Chen et al. [6] studied the existence of ground state solutions for Eq. (1.6), respectively. In [5] and [6], there are difficulties lie in two aspects. One is that the nonlinearity of equation is nonlocal and the other is that the energy functional is not well defined. Both of them adopted Liu and Wang's [25] approach, considering the change of variables $f : \mathbb{R} \to \mathbb{R}$ given by

$$f'(t) = \frac{1}{\sqrt{1+2f^2(t)}},\ \ -f(t) = f(-t).$$

By the change of $u = f(v)$ of variable, Eq. (1.6) is transformed into a semilinear problem

$$-\Delta v + V(x)f(v)f'(v) = (I_\alpha * |f(v)|^p)|f(v)|^{p-2}f(v)f'(v),\ \ x \in \mathbb{R}^N.$$

With this method, the two difficulties mentioned above can be solved.

There's also a lot of work focusing on semilinear problems, and we can refer to [26,27,35] and references therein. In [35], Tang and Chen considered the following singularly perturbed problem:

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{-\alpha}(I_\alpha * F(u))f(u),\ \ x \in \mathbb{R}^N. \tag{1.7}$$

The authors proved the existence of a ground state solution of Eq. (1.7) when ε was taken at different values and the nonlinearity f satisfied some suitable conditions, as well as the potential V. In particular, when $\varepsilon = 1$, the result is the improvement and expansion of Moroz and Van Schaftigen [27]'s conclusions. Moroz and Van Schaftigen [27] was the earliest one who proved the existence of a least energy to semilinear problems. On the basis of Jeanjean [10]'s method, Moroz and Van Schaftigen [27] constructed a (PS)-sequence that meets asymptotically the Pohožaev identity. With the related information to the Pohožaev identity, they can ensure the boundedness of (PS) sequence. And then a concentration compactness argument is used to solve the problem caused by lack of Sobolev embeddings. However, the approach proposed in [27] is only suitable for autonomous equations and useless for non-autonomous equations. Hence, on this basis, Tang and Chen [35] used Pohožaev manifold to study the existence of ground state solutions of non-autonomous equations.

As far as we know, there are few articles paying attention to Choquard type non-linearity for generalized quasilinear Schrödinger equations. Hence, motivated by the previously mentioned papers ([5,6,35]), we shall study the existence of positive solutions and ground state solutions for Eq. (1.1) using a change of variables and variational argument. Next, we give the following conditions on V:

(V_1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $0 < V_0 := \inf_{x\in\mathbb{R}^N} V(x)$, for all $x \in \mathbb{R}^N$;
(V_2) $V(x) \leq V_\infty$, for all $x \in \mathbb{R}^N$;
(V_3) $V(x) = V(|x|)$, for all $x \in \mathbb{R}^N$;
(V_4) $V(x) \in C^1(\mathbb{R}^N, \mathbb{R})$, there exist a constant $\theta \in (0, 1)$ and $L \geq 0$ such that

$$(\nabla V(x) \cdot x) \leq \begin{cases} \frac{(N-2)^2}{2|x|^2}, & \text{if} \quad 0 < |x| < L, \\ \alpha\theta V(x), & \text{if} \quad |x| \geq L; \end{cases}$$

(V_5) $V(x)$ is 1-periodic in each variable of $x_1, \cdots, x_N$.

In addition, we assume that the nonlinear term $g \in C^1(\mathbb{R}, (0, +\infty))$ is even, $g(0) = 1$, non-decreasing in $[0, +\infty)$ and satisfies

$$g_\infty := \lim_{t\to\infty} \frac{g(t)}{t} \in (0, \infty),$$

and

$$\beta := \sup_{t\in\mathbb{R}} \frac{tg'(t)}{g(t)} \leq 1.$$

The Eq. (1.1) is the Euler–Lagrange equation of the energy functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (g^2(u)|\nabla u|^2 + V(x)u^2) - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u^+|^p)|u^+|^p,$$

where $u^+ = \max\{u, 0\}$. Unfortunately, the energy functional J is not well defined for all $u \in H^1(\mathbb{R}^N)$ if $N \geq 3$. To solve this problem, we use the change of variables $v = G(u)$, where $G(t) := \int_0^t g(\tau)\mathrm{d}\tau$, then Eq. (1.1) will become

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = (I_\alpha * |G^{-1}(v)|^p)\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))}, \ x \in \mathbb{R}^N, \tag{1.8}$$

and $J(u)$ can be reduced to

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)[G^{-1}(v)]^2) - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |G^{-1}(v^+)|^p)|G^{-1}(v^+)|^p.$$

Obviously, the energy functional I is well defined in $H^1(\mathbb{R}^N)$. It is easy to see that if $v \in H^1(\mathbb{R}^N)$ is a critical point of I,

$$\begin{aligned}\langle I'(v), \varphi\rangle &= \int_{\mathbb{R}^N} \nabla v \nabla \varphi + \int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi \\ &\quad - \int_{\mathbb{R}^N} (I_\alpha * |G^{-1}(v^+)|^p)\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\varphi,\end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, then v is a weak solution of (1.8), that is, $u = G^{-1}(v)$ is a weak solution of (1.1).

Remark 1.1 Let

$$g(s) = \begin{cases} \sqrt{1+s^2}, & \text{if } \ 0 \le s \le 1, \\ \frac{\sqrt{2}}{2}(s+1), & \text{if } \ s > 1, \\ g(-s), & \text{if } \ s < 0. \end{cases}$$

or

$$g(s) = \sqrt{1+ks^2}, \ \ k > 0.$$

By a simple computation, it is obvious that the functions mentioned above satisfy the above conditions for g.

The main result of this paper is stated as follows:

Theorem 1.2 *Suppose that $N \ge 3$, $\frac{2(N+\alpha)}{N} < p < \frac{2(N+\alpha)}{N-2}$ and the potential function V satisfies (V_1), (V_2) and (V_5). Then, Eq. (1.1) possesses a positive solution $u \in H^1(\mathbb{R}^N)$.*

Theorem 1.3 *Assume that $N \ge 3$, $\frac{2(N+\alpha)}{N} < p < \frac{2(N+\alpha)}{N-2}$ and the potential function V satisfies $(V_1) - (V_4)$. Then Eq. (1.1) possesses a ground state solution.*

Notations In this paper, we need the following notations:

- let $D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ with the norm $\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^N} |\nabla u|^2$;
- $H^1(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ with the norm $\|u\|^2 := \|u\|^2_{H^1} = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)$;
- the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for $s \in [2, 2^*]$ and $H^1_r(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $s \in (2, 2^*)$;

- $H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2Nq}{N+\alpha}}(\mathbb{R}^N)$ if and only if $\frac{N+\alpha}{N} \le q \le \frac{N+\alpha}{N-2}$;
- $L^p(\mathbb{R}^N)$ denotes that the usual Lebesgue space with norm $\|u\|_p = (\int_{\mathbb{R}^N} |u|^p)^{\frac{1}{p}}$, where $1 \le p < \infty$;
- $\int_{\mathbb{R}^N} \clubsuit$ denotes $\int_{\mathbb{R}^N} \clubsuit \, \mathrm{d}x$;
- we use C or C_i to denote various positive constants in context.

The outline of the paper is as follows: in Sect. 2, we prove Theorem 1.2 by using the mountain pass theorem. In Sect. 3, we give the proof of Theorem 1.3.

2 Proof of Theorem 1.2

As quoted in the introduction, Eq. (1.1) is formally the Euler–Lagrange equation associated with the functional

$$u \mapsto \frac{1}{2}\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u^+|^p)|u^+|^p.$$

Since it is not well defined in $H^1(\mathbb{R}^N)$, we shall follow [32] and use the change of variables $v = G(u)$, where the function G is defined as $G(t) := \int_0^t g(\tau)d\tau$. Next, we list some of the important properties of function G^{-1}.

Lemma 2.1 [11] *The function $G^{-1} \in C^2(\mathbb{R}, \mathbb{R})$ satisfies the following properties:*

(g_1) G^{-1} *is increasing;*
(g_2) $0 < (G^{-1})'(t) = \frac{1}{g(G^{-1}(t))} \le 1$, *for all $t \in \mathbb{R}$;*
(g_3) $|G^{-1}(t)| \le |t|$, *for all $t \in \mathbb{R}$;*
(g_4) $\lim_{t\to 0} \frac{G^{-1}(t)}{t} = 1$;
(g_5) $\lim_{t\to\pm\infty} \frac{G^{-1}(t)}{g(G^{-1}(t))} = \pm\frac{1}{g_\infty}$;
(g_6) $1 \le \frac{tg(t)}{G(t)} \le 2$ *and* $1 \le \frac{G^{-1}(t)g(G^{-1}(t))}{t} \le 2$, *for all $t \ne 0$;*
(g_7) $\frac{G^{-1}(t)}{\sqrt{t}}$ *is non-decreasing in* $(0, +\infty)$ *and* $|G^{-1}(t)| \le (2/g_\infty)^{1/2}\sqrt{|t|}$, *for all $t \in \mathbb{R}$;*
(g_8) $|G^{-1}(t)| \ge \begin{cases} G^{-1}(1)|t|, & |t| \le 1; \\ G^{-1}(1)\sqrt{|t|}, & |t| \ge 1; \end{cases}$
(g_9) $\frac{t}{g(t)}$ *is increasing and* $|\frac{t}{g(t)}| \le \frac{1}{g_\infty}$, *for all $t \in \mathbb{R}$;*
(g_{10}) *the function* $[G^{-1}(t)]^2$ *is convex. In particular,* $[G^{-1}(st)]^2 \le s[G^{-1}(t)]^2$, *for all $t \in \mathbb{R}$, $s \in [0, 1]$;*
(g_{11}) $[G^{-1}(st)]^2 \le s^2[G^{-1}(t)]^2$, *for all $t \in \mathbb{R}$, $s \ge 1$;*
(g_{12}) $[G^{-1}(s-t)]^2 \le 4\left([G^{-1}(s)]^2 + [G^{-1}(t)]^2\right)$.

Lemma 2.2 [23] *(*Hardy-Littlewood-Sobolev inequality*) Let $r, s > 1$ and $0 < \alpha < N$ be such that*

$$\frac{1}{r} + \frac{1}{s} - \frac{\alpha}{N} = 1.$$

Where $f \in L^r(\mathbb{R}^N)$ and $h \in L^s(\mathbb{R}^N)$, there exists a constant C, independent of f, h, such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)h(y)}{|x-y|^{N-\alpha}} \leq C|f|_r|h|_s.$$

Next, we prove that the functional I exhibits the mountain pass geometry.

Lemma 2.3 *There exist $C_1 > 0$, $\rho_1 > 0$ such that*

$$\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V(x)[G^{-1}(v)]^2\right) \geq C_1\|v\|^2, \ \ \|v\| \leq \rho_1. \tag{2.1}$$

Proof Similar to [12], by contradiction, assume that (2.1) is not true, then for all n, there exists a sequence $\{u_n\} \neq 0$ such that $\|u_n\| \leq \frac{1}{n}$, we have

$$\int_{\mathbb{R}^N}\left(|\nabla u_n|^2 + V(x)[G^{-1}(u_n)]^2\right) \leq \frac{1}{n}\|u_n\|^2,$$

which can deduce that

$$\int_{\mathbb{R}^N}\frac{|\nabla u_n|^2}{\|u_n\|^2} + \int_{\mathbb{R}^N}V(x)\frac{[G^{-1}(u_n)]^2}{\|u_n\|^2} \leq \frac{1}{n},$$

let $v_n = \frac{u_n}{\|u_n\|}$, we can get

$$\int_{\mathbb{R}^N}(|\nabla v_n|^2 + V(x)v_n^2) + \int_{\mathbb{R}^N}V(x)\left(\frac{[G^{-1}(u_n)]^2}{u_n^2} - 1\right)v_n^2 \leq \frac{1}{n}.$$

Since as $n \to \infty$,

$$\begin{aligned}
&u_n \to 0 \ a.e.\ x \in \mathbb{R}^N,\\
&u_n \to 0 \text{ in } L^2(\mathbb{R}^N),\\
&meas\{x \in \mathbb{R}^N : |u_n(x)| > \varepsilon\} \to 0 \text{ for all } \varepsilon > 0.
\end{aligned}$$

Hence, by the Hölder inequality,

$$\begin{aligned}
\int_{|u_n|>\varepsilon} v_n^2 &\leq \left(\int_{|u_n|>\varepsilon}(v_n^2)^{\frac{r}{2}}\right)^{\frac{2}{r}}\left(\int_{|u_n|>\varepsilon}1\right)^{1-\frac{2}{r}}\\
&= (meas\{x \in \mathbb{R}^N : |u_n(x)| > \varepsilon\})^{1-\frac{2}{r}} \cdot \|v_n\|_r^2 \to 0,
\end{aligned} \tag{2.2}$$

where $N \geq 3$, $r = 2^*$.

Since $V(x)$ and $\{v_n\}$ are bounded, it follows from (g_4) that

$$\int_{\mathbb{R}^N} V(x)\left(\frac{[G^{-1}(u_n)]^2}{u_n^2}-1\right)v_n^2 = \int_{\mathbb{R}^N} V(x)\left(\frac{G^{-1}(u_n)}{u_n}-1\right)\left(\frac{G^{-1}(u_n)}{u_n}+1\right)v_n^2 \\ \to 0, \text{ as } u_n \to 0.$$

So $\|v_n\| = 1$ and $v_n \to 0$ in $H^1(\mathbb{R}^N)$, a contradiction. This completes the proof of Lemma 2.3. □

Lemma 2.4 *There exist* $\rho_0, \alpha > 0$ *such that*

$$I(v) \geq \alpha, \ \ for\ all\ v \in \left\{v \in H^1(\mathbb{R}^N) : \|v\| = \rho_0\right\}.$$

Proof Notice that $\frac{Np}{N+\alpha} \in (2, 2^*)$. By (g_7), (2.1), Hardy-Littlewood-Sobolev inequality and the Sobolev embedding theorem, we get

$$\begin{aligned} I(v) &\geq \frac{C_1}{2}\|v\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N}\left(I_\alpha * |G^{-1}(v^+)|^p\right)\left|G^{-1}(v^+)\right|^p \\ &\geq \frac{C_1}{2}\|v\|^2 - C_2\left(\int_{\mathbb{R}^N} I_\alpha * |v|^{\frac{p}{2}}\right)|v|^{\frac{p}{2}} \\ &\geq \frac{C_1}{2}\|v\|^2 - C_2\left(\int_{\mathbb{R}^N} |v|^{\frac{Np}{N+\alpha}}\right)^{\frac{N+\alpha}{N}} \\ &\geq \frac{C_1}{2}\|v\|^2 - C_3\|v\|^p \\ &\geq \|v\|^2\left(\frac{C_1}{2} - C_3\|v\|^{p-2}\right). \end{aligned}$$

Choosing ρ_0 small enough, we get the proof. □

Lemma 2.5 *There exists* $v_0 \in H^1(\mathbb{R}^N)$ *such that* $\|v_0\| > \rho_0$ *and* $I(v_0) < 0$.

Proof By (g_6), $\frac{G^{-1}(t)}{t}$ is decreasing for $t > 0$. Consider $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \phi(x) \leq 1$, $\phi(x) \leq 1$ for $|x| \leq 1$, $\phi(x) = 0$ for $|x| \geq 2$. We have

$$G^{-1}(t\phi(x)) \geq G^{-1}(t)\phi(x),$$

for any $x \in \mathbb{R}^N$, $t > 0$. Using (g_3), we get

$$
\begin{aligned}
I(t\phi) &= \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla\phi|^2+\frac{1}{2}\int_{\mathbb{R}^N}V(x)[G^{-1}(t\phi)]^2\\
&\quad -\frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha*|G^{-1}(t\phi)|^p)|G^{-1}(t\phi)|^p\\
&\le \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla\phi|^2+\frac{t^2}{2}\int_{\mathbb{R}^N}V(x)\phi^2-\frac{[G^{-1}(t)]^{2p}}{2p}\int_{\mathbb{R}^N}(I_\alpha*|\phi|^p)|\phi|^p\\
&\le \frac{t^2}{2}\left(C_1\|\phi\|^2-C_2\frac{[G^{-1}(t)]^4}{t^2}\cdot[G^{-1}(t)]^{2p-4}\right).
\end{aligned}
$$

By $p>2$ and (g_8), we deduce that $I(t_0\phi)<0$ and $t_0\|\phi\|>\rho_0$ for t_0 large enough. Set $v_0=t_0\phi$, hence v_0 is required. □

By [37, Theorem 6.3], combining Lemma 2.4 and Lemma 2.5, there exists a sequence $\{v_n\}\subset H^1(\mathbb{R}^N)$ satisfying that $I(v_n)\to c$ and $\|I'(v_n)\|(\|v_n\|+1)\to 0$, which is called Cerami sequence.

Lemma 2.6 *All Cerami sequences for I at the level $c>0$ are bounded in $H^1(\mathbb{R}^N)$.*

Proof Let $\{v_n\}\subset H^1(\mathbb{R}^N)$ be a Cerami sequence at the level c. Set $\omega_n := G^{-1}(v_n)g(G^{-1}(v_n))$. It follows from (g_2) and (g_6) that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}|\omega_n|^2\le 4\int_{\mathbb{R}^N}|v_n|^2,\\
&\int_{\mathbb{R}^N}|\nabla\omega_n|^2=\int_{\mathbb{R}^N}\left[1+\frac{G^{-1}(v_n)g'(G^{-1}(v_n))}{g(G^{-1}(v_n))}\right]^2|\nabla v_n|^2\le 4\int_{\mathbb{R}^N}|\nabla v_n|^2,
\end{aligned}
$$

and

$$
\left|\langle I'(v_n),\omega_n\rangle\right|\le C\left\|I'(v_n)\right\|(\|v_n\|+1)\to 0,\ \ as\ n\to\infty.
$$

It follows that $\{\omega_n\}\subset H^1(\mathbb{R}^N)$ is bounded. Therefore,

$$
\begin{aligned}
c+o_n(1) &\ge I(v_n)-\frac{1}{2p}\langle I'(v_n),\omega_n\rangle\\
&=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2+\frac{1}{2}\int_{\mathbb{R}^N}V(x)[G^{-1}(v_n)]^2\\
&\quad -\frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha*|G^{-1}(v_n^+)|^p)|G^{-1}(v_n^+)|^p\\
&-\frac{1}{2p}\int_{\mathbb{R}^N}\left[1+\frac{G^{-1}(v_n)g'(G^{-1}(v_n))}{g(G^{-1}(v_n))}\right]^2|\nabla v_n|^2\\
&-\frac{1}{2p}\int_{\mathbb{R}^N}V(x)[G^{-1}(v_n)]^2\\
&+\frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha*|G^{-1}(v_n^+)|^p)|G^{-1}(v_n^+)|^p
\end{aligned}
$$

$$\geq \left(\frac{1}{2}-\frac{1}{p}\right)\left(\int_{\mathbb{R}^N}|\nabla v_n|^2+\int_{\mathbb{R}^N}V(x)[G^{-1}(v_n)]^2\right).$$

Since $p > 2$, the sequence $\{\int_{\mathbb{R}^N}|\nabla v_n|^2+\int_{\mathbb{R}^N}V(x)[G^{-1}(v_n)]^2\}$ is bounded. Obviously, both $\int_{\mathbb{R}^N}|\nabla v_n|^2$ and $\int_{\mathbb{R}^N}V(x)[G^{-1}(v_n)]^2$ are bounded. By the Sobolev embedding theorem and (g_8), we have

$$\begin{aligned}
\int_{\mathbb{R}^N}|v_n|^2 &= \int_{\{|v|\leq 1\}}|v_n|^2+\int_{\{|v|>1\}}|v_n|^2\\
&\leq C_1\int_{\{|v|\leq 1\}}|G^{-1}(v_n)|^2+\left(\int_{\{|v|>1\}}|v_n|\right)^{\theta}\left(\int_{\{|v|>1\}}|v_n|^{2^*}\right)^{1-\theta}\\
&\leq C_1\int_{\mathbb{R}^N}|G^{-1}(v_n)|^2+\left(\int_{\{|v|>1\}}[G^{-1}(v_n)]^2\right)^{\theta}\left(\int_{\mathbb{R}^N}|v_n|^{2^*}\right)^{1-\theta}\\
&\leq C_2\int_{\mathbb{R}^N}V(x)[G^{-1}(v_n)]^2+C_3\left(\int_{\mathbb{R}^N}V(x)[G^{-1}(v_n)]^2\right)^{\theta}\\
&\quad\left(\int_{\mathbb{R}^N}|\nabla v_n|^2\right)^{(1-\theta)\cdot\frac{2^*}{2}}\leq C,
\end{aligned}$$

where $\theta=\frac{2^*-2}{2^*-1}$. Hence $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. □

In the following, let us assume that $\{v_n\}\subset H^1(\mathbb{R}^N)$ is a Cerami sequence for I at the level of $c > 0$. By the preceding lemma, $\{v_n\}$ is bounded. Hence, going if necessary to a subsequence, there exists $v\in H^1(\mathbb{R}^N)$ such that $v_n\rightharpoonup v\in H^1(\mathbb{R}^N)$, $v_n(x)\to v(x)$ a.e. $x\in\mathbb{R}^N$ and $v_n\to v$ in $L^q_{loc}(\mathbb{R}^N)$ for all $q\in(2,2^*)$. Then, we have the following Lemmas 2.7 and 2.8.

Lemma 2.7 *Up to a subsequence, there exist $R,\ \beta > 0$ and $\{x_n\}\subset\mathbb{Z}^N$ such that*

$$\liminf_{n\to\infty}\int_{B_R(x_n)}|v_n|^2\geq\beta.$$

Proof If Lemma 2.7 is false, then it follows from the [36, Lemma 1.21] that, up to a subsequence,

$$v_n\to 0\ \text{ in } L^s(\mathbb{R}^N),\ \ s\in(2,2^*).$$

Hence,

$$\int_{\mathbb{R}^N}\left(I_\alpha*|G^{-1}(v_n^+)|^p\right)|G^{-1}(v_n^+)|^p\leq C\left(\int_{\mathbb{R}^N}|v_n|^{\frac{pr}{2}}\right)^{\frac{2}{r}}\to 0,$$

where $\frac{2}{r}-\frac{\alpha}{N}=1$. Since $G^{-1}(v_n)g(G^{-1}(v_n))$ is bounded in $H^1(\mathbb{R}^N)$ and $\|I'(v_n)\|\to 0$,

$$\begin{aligned}\left\langle I'(v_n),G^{-1}(v_n)g(G^{-1}(v_n))\right\rangle=&\int_{\mathbb{R}^N}\left(1+\frac{G^{-1}(v_n)g'(G^{-1}(v_n))}{g(G^{-1}(v_n))}\right)|\nabla v_n|^2\\&+\int_{\mathbb{R}^N}V(x)[G^{-1}(v_n)]^2\\&-\int_{\mathbb{R}^N}(I_\alpha*|G^{-1}(v_n^+)|^p)|G^{-1}(v_n^+)|^p\to 0.\end{aligned}$$

Since $g\in C^1(\mathbb{R},(0,+\infty))$ is even, $g(0)=1$, non-decreasing in $[0,+\infty)$, it is easy to check that $\frac{G^{-1}(v_n)g'(G^{-1}(v_n))}{g(G^{-1}(v_n))}\geq 0$. Then, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla v_n|^2+\int_{\mathbb{R}^N}V(x)[G^{-1}(v_n)]^2\\&\quad\leq\int_{\mathbb{R}^N}\left(1+\frac{G^{-1}(v_n)g'(G^{-1}(v_n))}{g(G^{-1}(v_n))}\right)|\nabla v_n|^2+\int_{\mathbb{R}^N}V(x)[G^{-1}(v_n)]^2\to 0.\end{aligned}$$

It follows that

$$\begin{aligned}c+o_n(1)=I(v_n)\ =&\frac{1}{2}\left(\int_{\mathbb{R}^N}|\nabla v_n|^2+V(x)[G^{-1}(v_n)]^2\right)\\&-\frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha*|G^{-1}(v_n^+)|^p)|G^{-1}(v_n^+)|^p\to 0,\end{aligned}$$

which is a contradiction. The proof is completed. □

Lemma 2.8 $\langle I'(v),\varphi\rangle=0$ *for any* $\varphi\in C_0^\infty(\mathbb{R}^N)$.

Proof For any $\varphi\in C_0^\infty(\mathbb{R}^N)$, the support of φ is contained in $B_{R_0}(0)$ for some $R_0>0$. Hence

$$\begin{aligned}&\left|\langle I'(v_n)-I'(v),\varphi\rangle\right|\leq\left|\int_{\mathbb{R}^N}\nabla(v_n-v)\nabla\varphi\right|\\&\quad+\left|\int_{\mathbb{R}^N}V(x)\left(\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}-\frac{G^{-1}(v)}{g(G^{-1}(v))}\right)\varphi\right|\\&\quad+\left|\int_{\mathbb{R}^N}\left[(I_\alpha*|G^{-1}(v_n^+)|^p)\frac{|G^{-1}(v_n^+)|^{p-1}}{g(G^{-1}(v_n^+))}-(I_\alpha*|G^{-1}(v^+)|^p)\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\right]\varphi\right|\\&\quad:=I_1+I_2+I_3.\end{aligned}$$

For $I_1:=\left|\int_{\mathbb{R}^N}\nabla(v_n-v)\nabla\varphi\right|$, since $v_n\rightharpoonup v$ in $H^1(\mathbb{R}^N)$, $I_1\to 0$ as $n\to\infty$.

For $I_2 := \left|\int_{\mathbb{R}^N} V(x)\left(\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))}\right)\varphi\right|$, by (g_2) and (g_3), we have

$$\left|\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))}\right|^2 \le 2\left(\left|\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\right|^2 + \left|\frac{G^{-1}(v)}{g(G^{-1}(v))}\right|^2\right)$$
$$\le 2(|v_n|^2 + |v|^2).$$

By $v_n \to v$ in $L^2_{loc}(\mathbb{R}^N)$ and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_{B_{R_0}(0)}\left|\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))}\right|^2 = 0.$$

Using (V_2) and the Hölder inequality, we have

$$I_2 \le V_\infty \int_{B_{R_0}(0)}\left|\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))}\right||\varphi|$$
$$\le V_\infty\left(\int_{B_{R_0}(0)}\left|\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))}\right|^2\right)^{\frac{1}{2}}\left(\int_{B_{R_0}(0)}|\varphi|^2\right)^{\frac{1}{2}} \to 0,$$

as $n \to \infty$. Moreover,

$$I_3 := \left|\int_{\mathbb{R}^N}\left[(I_\alpha * |G^{-1}(v_n^+)|^p)\frac{|G^{-1}(v_n^+)|^{p-1}}{g(G^{-1}(v_n^+))} - (I_\alpha * |G^{-1}(v^+)|^p)\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\right]\varphi\right|$$
$$\le \int_{\mathbb{R}^N}(I_\alpha * |G^{-1}(v_n^+)|^p)\left|\frac{|G^{-1}(v_n^+)|^{p-1}}{g(G^{-1}(v_n^+))} - \frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\right||\varphi|$$
$$+\left|\int_{\mathbb{R}^N}(I_\alpha * |G^{-1}(v_n^+)|^p)\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\varphi - \int_{\mathbb{R}^N}(I_\alpha * |G^{-1}(v^+)|^p)\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\varphi\right|$$
$$:= J_1 + J_2.$$

For $r = \frac{2N}{N+\alpha}$, by (g_7), (g_9),

$$\left|\frac{|G^{-1}(v_n^+)|^{p-1}}{g(G^{-1}(v_n^+))} - \frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\right|^{\frac{pr}{p-2}}$$
$$\le C_1\left(\left|\frac{|G^{-1}(v_n^+)|^{p-2}G^{-1}(v_n^+)}{g(G^{-1}(v_n^+))}\right|^{\frac{pr}{p-2}} + \left|\frac{|G^{-1}(v^+)|^{p-2}G^{-1}(v^+)}{g(G^{-1}(v^+))}\right|^{\frac{pr}{p-2}}\right)$$
$$\le C_2\left(\left|[G^{-1}(v_n^+)]^{p-2}\right|^{\frac{pr}{p-2}} + \left|[G^{-1}(v^+)]^{p-2}\right|^{\frac{pr}{p-2}}\right)$$
$$\le C_3\left(|v_n|^{\frac{pr}{2}} + |v|^{\frac{pr}{2}}\right).$$

Since $\frac{2(N+\alpha)}{N} < p < \frac{2(N+\alpha)}{N-2}$, $\frac{pr}{2} \in (2, 2^*)$. By $v_n \to v$ in $L^{\frac{pr}{2}}_{loc}(\mathbb{R}^N)$ and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_{B_{R_0}(0)}\left|\frac{|G^{-1}(v_n^+)|^{p-1}}{g(G^{-1}(v_n^+))}-\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\right|^{\frac{pr}{p-2}}=0.$$

By the boundedness of $\{v_n\}$, $\varphi \equiv 0$ on $B^c_{R_0}(0)$, the Hölder inequality and Hardy-Littlewood-Sobolev inequality, take $n \to \infty$,

$$\begin{aligned}
J_1 &= \int_{\mathbb{R}^N}(I_\alpha * |G^{-1}(v_n^+)|^p)\left|\frac{|G^{-1}(v_n^+)|^{p-1}}{g(G^{-1}(v_n^+))}-\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\right||\varphi|\\
&\le C_1\left(\int_{\mathbb{R}^N}|v_n|^{\frac{pr}{2}}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\left|\frac{|G^{-1}(v_n^+)|^{p-1}}{g(G^{-1}(v_n^+))}-\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\right|^r|\varphi|^r\right)^{\frac{1}{r}}\\
&= C_1\left(\int_{\mathbb{R}^N}|v_n|^{\frac{pr}{2}}\right)^{\frac{1}{2}}\left(\int_{B_{R_0}(0)}\left|\frac{|G^{-1}(v_n^+)|^{p-1}}{g(G^{-1}(v_n^+))}-\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\right|^r|\varphi|^r\right.\\
&\quad\left.+\int_{B^c_{R_0}(0)}\left|\frac{|G^{-1}(v_n^+)|^{p-1}}{g(G^{-1}(v_n^+))}-\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\right|^r|\varphi|^r\right)^{\frac{1}{r}}\\
&\le C_2\left(\int_{B_{R_0}(0)}\left|\frac{|G^{-1}(v_n^+)|^{p-1}}{g(G^{-1}(v_n^+))}-\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\right|^r|\varphi|^r\right)^{\frac{1}{r}}\\
&\le C_3\left(\int_{B_{R_0}(0)}\left|\frac{|G^{-1}(v_n^+)|^{p-1}}{g(G^{-1}(v_n^+))}-\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\right|^{\frac{pr}{p-2}}\right)^{\frac{p-2}{pr}}\left(\int_{B_{R_0}(0)}|\varphi|^{\frac{pr}{2}}\right)^{\frac{2}{pr}}\\
&\le C_4\left(\int_{B_{R_0}(0)}\left|\frac{|G^{-1}(v_n^+)|^{p-1}}{g(G^{-1}(v_n^+))}-\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\right|^{\frac{pr}{p-2}}\right)^{\frac{p-2}{pr}}\to 0,
\end{aligned}$$

where $r = \frac{2N}{N+\alpha}$. For $r = \frac{2N}{N+\alpha}$, by $\frac{2(N+\alpha)}{N} < p < \frac{2(N+\alpha)}{N-2}$, (g_9) and the Hölder inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^N}\left|\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\varphi\right|^r &\le C\int_{\mathbb{R}^N}|G^{-1}(v^+)|^{2\cdot\frac{p-2}{2}\cdot r}|\varphi|^r\\
&\le C\int_{\mathbb{R}^N}|v|^{\frac{(p-2)r}{2}}|\varphi|^r\\
&\le C\left(\int_{\mathbb{R}^N}|v|^{\frac{(p-2)r}{2}\cdot\frac{p}{p-2}}\right)^{\frac{p-2}{p}}\left(\int_{\mathbb{R}^N}|\varphi|^{\frac{pr}{2}}\right)^{\frac{2}{p}}\\
&= C\left(\int_{\mathbb{R}^N}|v|^{\frac{pr}{2}}\right)^{\frac{p-2}{p}}\left(\int_{\mathbb{R}^N}|\varphi|^{\frac{pr}{2}}\right)^{\frac{2}{p}}
\end{aligned}$$

$$= C\,|v|_{\frac{pr}{2}}^{\frac{(p-2)r}{2}}\,|\varphi|_{\frac{pr}{2}}^{r}\,.$$

It follows from $\frac{pr}{2} \in (2, 2^*)$ that $\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\varphi \in L^r(\mathbb{R}^N)$.

To prove $J_2 \to 0$, we use an argument which is partly an adaptation of the proof of [27, Proposition 2.2]. Set a linear functional

$$T(u) := \int_{\mathbb{R}^N} (I_\alpha * u)\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\varphi.$$

Then, by Hardy-Littlewood-Sobolev inequality, $T : L^r(\mathbb{R}^N) \to \mathbb{R}$, where $r = \frac{2N}{N+\alpha}$, is a continuous linear functional, that is,

$$|T(u)| \le C\left(\int_{\mathbb{R}^N} |u|^r\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}^N}\left|\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\varphi\right|^r\right)^{\frac{1}{r}}.$$

As $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $|G^{-1}(v_n^+)|^{pr} \le C|v_n|^{\frac{pr}{2}}$, the sequence $(|G^{-1}(v_n^+)|^p)$ is bounded in $L^r(\mathbb{R}^N)$. We may assume, going if necessary to a subsequence, $|G^{-1}(v_n^+)|^p \rightharpoonup |G^{-1}(v^+)|^p$ in $L^r(\mathbb{R}^N)$. Then $T\left(|G^{-1}(v_n^+)|^p\right) \to T\left(|G^{-1}(v^+)|^p\right)$ as $n \to \infty$, that is,

$$\begin{aligned} J_2 &= \left|\int_{\mathbb{R}^N} (I_\alpha * |G^{-1}(v_n^+)|^p)\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\varphi \right.\\ &\quad \left. - \int_{\mathbb{R}^N} (I_\alpha * |G^{-1}(v^+)|^p)\frac{|G^{-1}(v^+)|^{p-1}}{g(G^{-1}(v^+))}\varphi\right| \to 0. \end{aligned}$$

Therefore, $I_3 = J_1 + J_2 \to 0$ as $n \to \infty$. In a summary, up to a subsequence, we prove that $\langle I'(v_n) - I'(v), \varphi\rangle \to 0$ as $n \to \infty$. Since $\langle I'(v_n), \varphi\rangle \to 0$, we have

$$\langle I'(v), \varphi\rangle = 0.$$

The proof is completed. □

Proof of Theorem 1.2 As a consequence of Lemma 2.4 and 2.5, for the constant

$$c_0 = \inf_{r\in\Gamma}\sup_{t\in[0,1]} I(\gamma(t)) > 0,$$

where

$$\Gamma = \{\gamma \in C([0, 1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Hence, by [37, Theorem 6.3], there exists a Cerami sequence $\{v_n\}$ in $H^1(\mathbb{R}^N)$ at the level c_0, that is,

$$I(v_n) \to c_0 \text{ and } (1 + \|v_n\|)\|I'(v_n)\| \to 0, \text{ as } n \to \infty.$$

By Lemma 2.6, up to a sequence $\{v_n\}$ is bounded. Hence, up to a subsequence, one has $v_n \rightharpoonup v \in H^1(\mathbb{R}^N)$, $v_n(x) \to v(x)$ a.e. $x \in \mathbb{R}^N$ and $v_n \to v$ in $L^q_{loc}(\mathbb{R}^N)$ for all $q \in (2, 2^*)$.

By Lemma 2.7, up to a subsequence, there exist $R,\ \beta > 0$ and $\{y_n\} \subset \mathbb{Z}^N$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |v_n|^2 \geq \beta.$$

Define $\omega_n(x) = v_n(x + y_n)$ so that

$$\liminf_{n\to\infty} \int_{B_R(0)} |\omega_n|^2 \geq \beta > 0. \tag{2.3}$$

Since $V(x)$ is periodic in x, we have $\|\omega_n\| = \|v_n\|$ and

$$I(\omega_n) \to c_0 \text{ and } (1 + \|\omega_n\|)\|I'(\omega_n)\| \to 0, \text{ as } n \to \infty. \tag{2.4}$$

Up to a subsequence, one has $\omega_n \rightharpoonup \omega \in H^1(\mathbb{R}^N)$, $\omega_n(x) \to \omega(x)$ a.e. $x \in \mathbb{R}^N$ and $\omega_n \to \omega$ in $L^q_{loc}(\mathbb{R}^N)$ for all $q \in (2, 2^*)$. Hence, it follows from (2.3) ω is nontrivial. Similar to the proof of Lemma 2.8 and (2.4), we can obtain $I'(\omega) = 0$. This shows that $\omega \in H^1(\mathbb{R}^N)$ is a nontrivial, nonnegative, weak solution of (1.8). According to the strong maximum principle [13], $\omega > 0$ in $\mathbb{R}^N$. This completes the proof of Theorem 1.2. □

3 Proof of Theorem 1.3

In this section, we would like to complete the proof of Theorem 1.3.

Theorem 3.1 [17] *Let $(X, \|\cdot\|)$ be a Banach space and $\mathbb{I} \subset \mathbb{R}_+$ is an interval. Consider the following family of C^1-functionals on X:*

$$I_\lambda(v) = A(v) - \lambda B(v),\ \ \lambda \in \mathbb{I},$$

with B is nonnegative and either $A(v) \to +\infty$ or $B(v) \to +\infty$ as $\|v\| \to \infty$. Suppose that there are two points v_1, v_2 in X such that

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\}\ \ for\ all\ \lambda \in \mathbb{I},$$

where $\Gamma_\lambda = \{\gamma \in C([0, 1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}$. Then for almost every $\lambda \in \mathbb{I}$, there is a sequence $\{v_n\} \subset X$ such that

(i) *$\{v_n\}$ is bounded,*
(ii) *$I_\lambda(v_n) \to c_\lambda$,*
(iii) *$I'_\lambda(v_n) \to 0$ in the dual X^{-1} of X.*

Moreover, the map $\lambda \mapsto c_\lambda$ is non-increasing and continuous from the left.

Let $\mathbb{I} = [\frac{1}{2}, 1]$. We define the following energy functional

$$\begin{aligned} I_\lambda(v) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)v^2) \\ &-\lambda\int_{\mathbb{R}^N}\left(\frac{1}{2}V(x)(v^2 - [G^{-1}(v)]^2) + \frac{1}{2p}(I_\alpha * |G^{-1}(v)|^p)|G^{-1}(v)|^p\right), \end{aligned} \tag{3.1}$$

where $\lambda \in \mathbb{I}$. Then, let $A(v) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)v^2)$ and

$$B(v) = \int_{\mathbb{R}^N}\left(\frac{1}{2}V(x)(v^2 - [G^{-1}(v)]^2) + \frac{1}{2p}(I_\alpha * |G^{-1}(v)|^p)|G^{-1}(v)|^p\right).$$

Let $\|v\| \to \infty$, then $A(v) \to +\infty$. Moreover, $B(v) \geq 0$.

Similar to [26,36], we can get the following Pohožave type identity.

Lemma 3.2 *If $v \in H^1(\mathbb{R}^N)$ be a critical point of (3.1), then v satisfies*

$$\begin{aligned} \mathcal{P}_\lambda(v) := \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N}(\nabla V(x)\cdot x)[G^{-1}(v)]^2 + \frac{N}{2}\int_{\mathbb{R}^N}V(x)[G^{-1}(v)]^2 \\ - \frac{(N+\alpha)\lambda}{2p}\int_{\mathbb{R}^N}(I_\alpha * |G^{-1}(v)|^p)|G^{-1}(v)|^p = 0. \end{aligned}$$

Lemma 3.3 *Assume that (V_1)-(V_3) are satisfied.*

(i) *there exists $v \in H_r^1(\mathbb{R}^N)\backslash\{0\}$ such that $I_\lambda(v) < 0$ for all $\lambda \in \mathbb{I}$;*

(ii) *$c_\lambda = \inf\limits_{\gamma\in\Gamma_\lambda}\max\limits_{t\in[0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(0), I_\lambda(v)\}$ for all $\lambda \in \mathbb{I}$, where*

$$\Gamma_\lambda = \{\gamma \in C([0,1], H_r^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = v\}.$$

Proof (i) Let $v \in H_r^1(\mathbb{R}^N)\backslash\{0\}$ be fixed. For any $\lambda \in \mathbb{I} = [\frac{1}{2}, 1]$, one has

$$\begin{aligned} I_\lambda(v) &\leq I_{\frac{1}{2}}(v) \\ &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N}V(x)(v^2 - [G^{-1}(v)]^2) \\ &\quad + \frac{1}{4p}\int_{\mathbb{R}^N}(I_\alpha * |G^{-1}(v)|^p)|G^{-1}(v)|^p. \end{aligned}$$

As the proof of Lemma 2.5, we have

$$\begin{aligned} I_\lambda(t\phi) &\leq \frac{t^2}{2}\int_{\mathbb{R}^N}(|\nabla\phi|^2 + V(x)\phi^2) - \frac{1}{4p}\int_{\mathbb{R}^N}(I_\alpha * |G^{-1}(t\phi)|^p)|G^{-1}(t\phi)|^p \\ &\leq \frac{t^2}{2}\left[\int_{\mathbb{R}^N}(|\nabla\phi|^2 + V(x)\phi^2) - \frac{[G^{-1}(t)]^{2p-4}}{2p}\cdot\frac{[G^{-1}(t)]^4}{t^2}\int_{\mathbb{R}^N}(I_\alpha * |\phi|^p)|\phi|^p\right]. \end{aligned}$$

It follows that $I_\lambda(t\phi) \to -\infty$ as $t \to +\infty$. Thus, there exists a $t_0 > 0$ such that $I_\lambda(t_0\phi) < 0$. Then, taking $v_0 = t_0\phi$, we have $I_\lambda(v_0) < 0$ for all $\lambda \in \mathbb{I}$.

(ii) By Lemma 2.3 and 2.4, we can get

$$\begin{aligned} I_\lambda(v) &\geq \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)[G^{-1}(v)]^2) - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |G^{-1}(v)|^p)|G^{-1}(v)|^p \\ &\geq C(\|v\|^2 - \|v\|^p), \text{ for all } \|v\| \leq \rho_1. \end{aligned}$$

Since $p > 2$, we deduce that I_λ has a strict local minimum at 0 and hence $c_\lambda > 0$. □

By Theorem 3.1, it is easy to know that for a.e. $\lambda \in [\frac{1}{2}, 1]$, there exists a bounded sequence $\{v_n\} \subset H_r^1(\mathbb{R}^N)$ such that $I_\lambda(v_n) \to c_\lambda$ and $I'_\lambda(v_n) \to 0$, which is called (PS)-sequence.

Lemma 3.4 *If $\{v_n\} \subset H_r^1(\mathbb{R}^N)$ is the sequence obtained above, then for almost every $\lambda \in \mathbb{I}$, there exists $v_\lambda \in H_r^1(\mathbb{R}^N)\backslash\{0\}$ such that $I_\lambda(v_\lambda) = c_\lambda$ and $I'_\lambda(v_\lambda) = 0$.*

Proof Since $\{v_n\} \subset H^1(\mathbb{R}^N)$ is bounded, up to a subsequence, there exists $v_\lambda \in H_r^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v_\lambda$ in $H^1(\mathbb{R}^N)$, $v_n \to v_\lambda$ in $L^s(\mathbb{R}^N)$ for all $s \in (2, 2^*)$ and $v_n \to v_\lambda$ a.e. in $\mathbb{R}^N$. By the Lebesgue dominated convergence theorem, it is easy to check that $I'_\lambda(v_\lambda) = 0$. Next, let us first prove that there exists $C > 0$ such that

$$\int_{\mathbb{R}^N}\left[|\nabla(v_n - v_\lambda)|^2 + V(x)\left(\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v_\lambda)}{g(G^{-1}(v_\lambda))}\right)(v_n - v_\lambda)\right] \geq C\|v_n - v_\lambda\|^2. \tag{3.2}$$

Similar to [12,38], we assume $v_n \neq v_\lambda$ (otherwise the conclusion is trivial). Set

$$\omega_n = \frac{v_n - v_\lambda}{\|v_n - v_\lambda\|} \quad \text{and} \quad h_n = \frac{\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v_\lambda)}{g(G^{-1}(v_\lambda))}}{v_n - v_\lambda}$$

We argue by a contradiction and suppose v_n, v_λ may be found such that

$$\int_{\mathbb{R}^N}|\nabla\omega_n|^2 + V(x)h_n(x)\omega_n^2 \to 0.$$

Since

$$\frac{d}{dt}\left(\frac{G^{-1}(t)}{g(G^{-1}(t))}\right) = \frac{g(G^{-1}(t)) - G^{-1}(t)g'(G^{-1}(t))}{g^3(G^{-1}(t))} > 0,$$

$\frac{G^{-1}(t)}{g(G^{-1}(t))}$ is strictly increasing and for each $C > 0$, there exists $\delta > 0$ such that

$$\frac{d}{dt}\left(\frac{G^{-1}(t)}{g(G^{-1}(t))}\right) \geq \delta, \tag{3.3}$$

as $|t| \leq C$. It's easy to see that $h_n(x)$ is positive if $\omega_n(x) \neq 0$. Hence

$$\int_{\mathbb{R}^N} |\nabla \omega_n|^2 \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} V(x) h_n(x) \omega_n^2 \to 0.$$

Because $\|\omega_n\| = \int_{\mathbb{R}^N}(|\nabla\omega_n|^2 + V(x)\omega_n^2) = 1$, $\int_{\mathbb{R}^N} V(x)\omega_n^2 \to 1$. For a given $C_1 > 0$, let $A_n = \{x \in \mathbb{R}^N : |v_n(x)| \geq C_1 \text{ or } |v_\lambda(x)| \geq C_1\}$, $B_n = \mathbb{R}^N \backslash A_n$. Then for each $\varepsilon > 0$, C_1 may be chosen so that the measure $|A_n| \leq \varepsilon$. It follows from (3.3) and the Mean Value Theorem that

$$\delta \int_{B_n} V(x)\omega_n^2 \leq \int_{B_n} V(x) h_n(x) \omega_n^2 \to 0. \tag{3.4}$$

Choosing ε small enough and arguing as in (2.2) (with the same r), we have

$$\int_{A_n} V(x)\omega_n^2 \leq C_2 \varepsilon^{\frac{r-2}{r}} \leq \frac{1}{2}. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain

$$\int_{\mathbb{R}^N} V(x)\omega_n^2 = \int_{B_n} V(x)\omega_n^2 + \int_{A_n} V(x)\omega_n^2 \leq \frac{1}{2} + o(1),$$

a contradiction. The proof of (3.2) is completed.

Moreover, using Hardy-Littlewood-Sobolev inequality, (g_5), (g_7) and the Hölder inequality, we deduce that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} (I_\alpha * |G^{-1}(v_n)|^p) \frac{|G^{-1}(v_n)|^{p-2} G^{-1}(v_n)}{g(G^{-1}(v_n))} (v_n - v_\lambda) \right| \\
&\leq C \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{\frac{p}{2}}) |v_n|^{\frac{p}{2}-1} |v_n - v_\lambda| \\
&\leq C \left(\int_{\mathbb{R}^N} |v_n|^{\frac{p}{2} r} \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^N} |v_n|^{\frac{p-2}{2} r} |v_n - v_\lambda|^r \right)^{\frac{1}{r}} \\
&\leq C \left(\left(\int_{\mathbb{R}^N} |v_n|^{\frac{p-2}{2} r \cdot \frac{p}{p-2}} \right)^{\frac{p-2}{p}} \left(\int_{\mathbb{R}^N} |v_n - v_\lambda|^{r \cdot \frac{p}{2}} \right)^{\frac{2}{p}} \right)^{\frac{1}{r}} \\
&\leq C \left(\int_{\mathbb{R}^N} |v_n - v_\lambda|^{r \cdot \frac{p}{2}} \right)^{\frac{2}{pr}} \to 0, \quad \frac{2}{r} - \frac{\alpha}{N} = 1.
\end{aligned} \tag{3.6}$$

In the same way, we can prove that

$$\left| \int_{\mathbb{R}^N} (I_\alpha * |G^{-1}(v_\lambda)|^p) \frac{|G^{-1}(v_\lambda)|^{p-2} G^{-1}(v_\lambda)}{g(G^{-1}(v_\lambda))} (v_n - v_\lambda) \right| \to 0. \tag{3.7}$$

Thus, it follows from (3.2), (3.6), (3.7) that

$$\begin{aligned}0 \leftarrow &\langle I'_\lambda(v_n) - I'_\lambda(v_\lambda), v_n - v_\lambda\rangle \\ &= \int_{\mathbb{R}^N}\left[|\nabla(v_n - v_\lambda)|^2 + V(x)\left(\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v_\lambda)}{g(G^{-1}(v_\lambda))}\right)(v_n - v_\lambda)\right] \\ &\quad - \int_{\mathbb{R}^N}(I_\alpha * |G^{-1}(v_n)|^p)\frac{G^{-1}(v_n)|G^{-1}(v_n)|^{p-2}}{g(G^{-1}(v_n))}(v_n - v_\lambda) \\ &\geq C\|v_n - v_\lambda\|^2 + o_n(1),\end{aligned}$$

which implies $v_n \to v_\lambda$ in $H^1_r(\mathbb{R}^N)$. Thus, v_λ is a nontrivial critical point of I_λ with $I_\lambda(v_\lambda) = c_\lambda$. □

Lemma 3.5 *Assume that* (V_4) *hold. Then, we have the following inequality:*

$$\begin{aligned}&(\alpha+2)\int_{\mathbb{R}^N}|\nabla v_{\lambda_n}|^2 + \int_{\mathbb{R}^N}[\alpha V(x) - \nabla V(x)\cdot x][G^{-1}(v_{\lambda_n})]^2 \\ &\quad\geq \alpha\int_{\mathbb{R}^N}|\nabla v_{\lambda_n}|^2 + (1-\theta)\alpha\int_{\mathbb{R}^N}V(x)[G^{-1}(v_{\lambda_n})]^2.\end{aligned}$$

Proof By Hardy's inequality [1]

$$\int_{\mathbb{R}^N}|\nabla u|^2 \geq \frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{u^2}{|x|^2},$$

we deduce that

$$\begin{aligned}\frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{[G^{-1}(v_{\lambda_n})]^2}{|x|^2} &\leq \int_{\mathbb{R}^N}|\nabla(G^{-1}(v_{\lambda_n}))|^2 \\ &= \int_{\mathbb{R}^N}\frac{1}{g^2(G^{-1}(v_{\lambda_n}))}|\nabla v_{\lambda_n}|^2 \\ &\leq \int_{\mathbb{R}^N}|\nabla v_{\lambda_n}|^2.\end{aligned} \tag{3.8}$$

From (V_4), (3.8), we have

$$
\begin{aligned}
&(\alpha+2)\int_{\mathbb{R}^N}|\nabla v_{\lambda_n}|^2+\int_{\mathbb{R}^N}[\alpha V(x)-\nabla V(x)\cdot x][G^{-1}(v_{\lambda_n})]^2\\
&=(\alpha+2)\int_{\mathbb{R}^N}|\nabla v_{\lambda_n}|^2+\int_{0<|x|<L}[\alpha V(x)-\nabla V(x)\cdot x][G^{-1}(v_{\lambda_n})]^2\\
&\quad+\int_{|x|\geq L}[\alpha V(x)-\nabla V(x)\cdot x][G^{-1}(v_{\lambda_n})]^2\\
&\geq(\alpha+2)\int_{\mathbb{R}^N}|\nabla v_{\lambda_n}|^2+\int_{\mathbb{R}^N}\alpha V(x)[G^{-1}(v_{\lambda_n})]^2\\
&\quad-2\int_{\mathbb{R}^N}|\nabla v_{\lambda_n}|^2-\int_{\mathbb{R}^N}\alpha\theta V(x)[G^{-1}(v_{\lambda_n})]^2\\
&=\alpha\int_{\mathbb{R}^N}|\nabla v_{\lambda_n}|^2+(1-\theta)\alpha\int_{\mathbb{R}^N}V(x)[G^{-1}(v_{\lambda_n})]^2.
\end{aligned}
$$

The proof is completed. □

Proof of Theorem 1.3 At first, by Theorem 3.1, for a.e. $\lambda\in\mathbb{I}=[\frac{1}{2},1]$, there is a $v_\lambda\in H_r^1(\mathbb{R}^N)$ such that $v_n\rightharpoonup v_\lambda\neq 0$ in $H_r^1(\mathbb{R}^N)$, $I_\lambda(v_n)\to c_\lambda$ and $I'_\lambda(v_n)\to 0$. Then, by Lemma 3.4, we get $I_\lambda(v_\lambda)=c_\lambda$ and $I'_\lambda(v_\lambda)=0$. Thus, there exists $\{\lambda_n\}\subset[\frac{1}{2},1]$ such that $\lambda_n\to 1$, $v_{\lambda_n}\in H^1(\mathbb{R}^N)$, $I_{\lambda_n}(v_{\lambda_n})=c_{\lambda_n}$ and $I'_{\lambda_n}(v_{\lambda_n})=0$. Next, we prove that $\{v_{\lambda_n}\}$ is bounded in $H_r^1(\mathbb{R}^N)$. In fact, from Lemma 3.3, $I_{\lambda_n}(v_{\lambda_n})\leq c_{\frac{1}{2}}$ and $I'_{\lambda_n}(v_{\lambda_n})=0$, it follows that

$$
\begin{aligned}
c_{\frac{1}{2}}\geq I_{\lambda_n}(v_{\lambda_n})&=I_{\lambda_n}\left(v_{\lambda_n}-\frac{1}{N+\alpha}\mathcal{P}_{\lambda_n}(v_{\lambda_n})\right)\\
&=\frac{1}{2(N+\alpha)}\left((\alpha+2)\int_{\mathbb{R}^N}|\nabla v_{\lambda_n}|^2\right.\\
&\quad\left.+\int_{\mathbb{R}^N}[\alpha V(x)-\nabla V(x)\cdot x][G^{-1}(v_{\lambda_n})]^2\right). \qquad (3.9)
\end{aligned}
$$

By (3.9) and Lemma 3.5, we get

$$
c_{\frac{1}{2}}\geq\frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}|\nabla v_{\lambda_n}|^2+\frac{(1-\theta)\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}V(x)[G^{-1}(v_{\lambda_n})]^2. \qquad (3.10)
$$

By Sobolev inequality, (V_1) and (g_8), it follows that

$$
\int_{|v_{\lambda_n}|\leq 1}v_{\lambda_n}^2\leq\frac{1}{V_0}\int_{R^N}V(x)[G^{-1}(v_{\lambda_n})]^2,
$$

and

$$\int_{|v_{\lambda_n}|>1} v_{\lambda_n}^2 \leq \int_{|v_{\lambda_n}|>1} v_{\lambda_n}^{2^*} \leq C\left(\int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2\right)^{\frac{2^*}{2}}.$$

Therefore,

$$\int_{\mathbb{R}^N} v_{\lambda_n}^2 = \int_{|v_{\lambda_n}|\leq 1} v_{\lambda_n}^2 + \int_{|v_{\lambda_n}|>1} v_{\lambda_n}^2 \leq \frac{1}{V_0}\int_{\mathbb{R}^N} V(x)[G^{-1}(v_{\lambda_n})]^2 + C\left(\int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2\right)^{\frac{2^*}{2}}. \tag{3.11}$$

According to (3.10) and (3.11), we infer that there exists a $C > 0$ such that $\int_{\mathbb{R}^N} v_{\lambda_n}^2 \leq C$. Hence, there is a constant $C > 0$ independent of n such that $\|v_{\lambda_n}\|^2 = \int_{\mathbb{R}^N}(|\nabla v_{\lambda_n}|^2 + v_{\lambda_n}^2) \leq C$. Then, we can suppose that the limit of $I_{\lambda_n}(v_{\lambda_n})$ exists. By Theorem 3.1, we know that $\lambda \to c_\lambda$ is continuous from the left. Therefore, we can get $0 \leq \lim_{n\to\infty} I_{\lambda_n}(v_{\lambda_n}) \leq c_{\frac{1}{2}}$. Then, using the fact that

$$I(v_{\lambda_n}) = I_{\lambda_n}(v_{\lambda_n}) + \frac{\lambda_n - 1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |G^{-1}(v_{\lambda_n})|^p)|G^{-1}(v_{\lambda_n})|^p,$$

$$\langle I'(v_{\lambda_n}), \phi\rangle = \langle I'_{\lambda_n}(v_{\lambda_n}), \phi\rangle + (\lambda_n - 1)\int_{\mathbb{R}^N}(I_\alpha * |G^{-1}(v_{\lambda_n})|^p)\frac{|G^{-1}(v_{\lambda_n})|^{p-1}}{g(G^{-1}(v_{\lambda_n}))}\phi,$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$ and $\|v_{\lambda_n}\| \leq C$, it follows that $\lim_{n\to\infty} I(v_{\lambda_n}) = c_1$ and $\lim_{n\to\infty} I'(v_{\lambda_n}) = 0$. Up to a subsequence, there exists a subsequence $\{v_{\lambda_n}\}$ denoted by $\{v_n\}$ and $v_0 \in H_r^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v_0$ in $H_r^1(\mathbb{R}^N)$. Preceding the same method as Lemma 3.4, we can obtain the existence of a nontrivial solution v_0 for I and $I'(v_0) = 0$ and $I(v_0) = c_1$.

To seek ground state solutions, we need to define $m := \inf\{I(v) : v \neq 0, I'(v) = 0\}$. By Lemma 3.2, it follows that $\mathcal{P}(v) = \mathcal{P}_1(v) = 0$. From (3.10), we have $m \geq 0$. Let $\{v_n\}$ be a sequence such that $I'(v_n) = 0$ and $I(v_n) \to m$. Similar to the discussion in Lemma 3.4, we can prove that there exists $\overline{v} \in H_r^1(\mathbb{R}^N)$ such that $I'(\overline{v}) = 0$ and $I(\overline{v}) = m$, which shows that $\overline{u} = G^{-1}(\overline{v})$ is a ground state solution of (1.1). According to the strong maximum principle [13], $\overline{u} > 0$ in $\mathbb{R}^N$. Theorem 1.3 is proved. □

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